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Minimal reducible bounds for induced-hereditary properties☆

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Abstract

Let $(\mathbb{M}^{\mathrm{a}}, \subseteq)$ and $(\mathbb{L}^{\mathrm{a}}, \subseteq)$ be the lattices of additive induced-hereditary properties of graphs and additive hereditary properties of graphs, respectively. A property $\mathcal{R} \in \mathbb{M}^{\mathrm{a}}$ $(\in \mathbb{L}^{\mathrm{a}})$ is called a *minimal reducible bound* for a property $\mathcal{P} \in \mathbb{M}^{\mathrm{a}}$ $(\in \mathbb{L}^{\mathrm{a}})$ if in the interval $(\mathcal{P}, \mathcal{R})$ of the lattice $\mathbb{M}^{\mathrm{a}}$ $(\mathbb{L}^{\mathrm{a}})$ there are only irreducible properties. The set of all minimal reducible bounds of a property $\mathcal{P} \in \mathbb{M}^{\mathrm{a}}$ in the lattice $\mathbb{M}^{\mathrm{a}}$ we denote by $\boldsymbol{B}_M(\mathcal{P})$. Analogously, the set of all minimal reducible bounds of a property $\mathcal{P} \in \mathbb{L}^{\mathrm{a}}$ in $\mathbb{L}^{\mathrm{a}}$ is denoted by $\boldsymbol{B}_L(\mathcal{P})$.

We establish a method to determine minimal reducible bounds for additive degenerate induced-hereditary (hereditary) properties of graphs. We show that this method can be successfully used to determine already known minimal reducible bounds for k-degenerate graphs and outerplanar graphs in the lattice $\mathbb{L}^{\mathrm{a}}$. Moreover, in terms of this method we describe the sets of minimal reducible bounds for partial k-trees and the graphs with restricted order of components in $\mathbb{L}^{\mathrm{a}}$ and k-degenerate graphs in $\mathbb{M}^{\mathrm{a}}$.

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1. Introduction

The standard graph colouring problem deals with the assignment of colours to vertices of a graph G in such a way that two adjacent vertices must be distinguished by their colours. Generalized colourings can be obtained by replacing adjacency constraint by some other condition on colour classes (see e.g. [3,4,7,11,12,19]).

A convenient language that may be used for formulating problems of graph colouring in a general setting is the language of reducible hereditary properties. The concept of reducible hereditary properties was introduced in [8,14].

A *graph property* is any non-empty isomorphism-closed subclass of graphs. Since we have, in general, no reason to distinguish between isomorphic copies of a graph, we use the notation $\mathcal{I}$ to denote the set of all finite unlabelled loopless undirected graphs. Therefore, by saying that H is a subgraph of G, we mean that H is isomorphic to a subgraph of G. Similarly we shall count a graph G and its isomorphic images as one graph. If G belongs to a property $\mathcal{P} \subseteq \mathcal{I}$ then we also say that G has the property $\mathcal{P}$. A property $\mathcal{P} \neq \mathcal{I}$ is called *non-trivial*.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-*partition* of G is a partition $(V_1, V_2, \ldots, V_n)$ of the vertex set $V(G)$ such that the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$ for $i = 1, 2, \ldots, n$. If a graph G has a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition, then we say that G has property $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$. If $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n$ we simply write $\mathcal{P}^n$ instead of $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$.

Since it is very difficult to deal with properties in such a general setting, we need an additional reasonable requirement. It seems to be fruitful to consider some partial order $\preccurlyeq$ on the set $\mathcal{I}$, for example "to be a subgraph", "to be an induced subgraph", "to be a minor" etc. We say that a property $\mathcal{P}$ is $\preccurlyeq$-*hereditary* if $G \in \mathcal{P}$ implies that $H \in \mathcal{P}$, for all

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$H \preccurlyeq G$. In particular, we shall deal with $\subseteq$-*hereditary* (in short *hereditary*) and $\preccurlyeq$-*hereditary* (we use also the term *induced-hereditary*) properties of graphs, meaning those which are closed under taking subgraphs and induced subgraphs, respectively. It is easy to observe that hereditary properties are special examples of induced-hereditary properties. A number of problems refer to specific types of hereditary properties which are called *additive*. Those properties are closed under taking the disjoint union of graphs with the given property, i.e. a property $\mathscr{P}$ is additive if $G, H \in \mathscr{P}$ implies $G \cup H \in \mathscr{P}$.

The *chromatic number* of a property $\mathscr{P}$ is defined in the following way:

$$\chi(\mathscr{P}) = \min\{\chi(F) : F \notin \mathscr{P}\}.$$

It is obvious that $\chi(\mathscr{P})$ is always at least two provided P is induced-hereditary and additive. A property $\mathscr{P}$ is called *degenerate* if its chromatic number is two, i.e. there is at least one bipartite graph which does not belong to $\mathscr{P}$.

A *k-tree* is a graph defined inductively as follows: a clique of order k is a k-tree. If G is a k-tree and K is a clique of G of order k, then the graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a *partial k-tree*.

We list some degenerate additive hereditary properties, using the notation of [4]:

$$\mathscr{O} = \{G \in \mathscr{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},$$

$$\mathscr{O}_k = \{G \in \mathscr{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\},$$

$$\mathscr{D}_k = \{G \in \mathscr{I} : G \text{ is } k\text{-degenerate, i.e. } \delta(H) \leqslant k \text{ for each } H \subseteq G\},$$

$$\mathscr{S}_k = \{G \in \mathscr{I} : \text{the maximum degree of } G \text{ is at most } k\},$$

$$\mathscr{T}_2 = \{G \in \mathscr{I} : G \text{ is outerplanar}\},$$

$$\mathscr{P}T_k = \{G \in \mathscr{I} : G \text{ is a partial } k\text{-tree}\}.$$

The notation $G \xrightarrow{i} (F, H)$ means that for any 2-colouring of the vertex set of G either F is an induced subgraph of the graph induced by the first colour class or a copy of H is an induced subgraph of the second colour class. The notation $\mathscr{P} \xrightarrow{i} (\mathscr{Q}_1, \mathscr{Q}_2)$ is used to express that for any graph G_1 with the property $\mathscr{Q}_1$ and for any graph G_2 with the property $\mathscr{Q}_2$ there exists a graph G with property $\mathscr{P}$ such that $G \xrightarrow{i} (G_1, G_2)$. If we consider subgraphs instead of induced subgraphs we use the notation $G \to (F, H)$ and $\mathscr{P} \to (\mathscr{Q}_1, \mathscr{Q}_2)$.

The set of all additive hereditary properties ordered by set inclusion forms a complete, algebraic and distributive lattice. We shall denote it by $\mathbb{L}^a$. The set of all induced-hereditary properties ordered by set inclusion forms a complete, algebraic and distributive lattice as well and it will be denoted by $\mathbb{M}^a$. Moreover $\mathbb{L}^a$ is a sublattice of $\mathbb{M}^a$. For much more details, many applications and open problems concerning hereditary and induced-hereditary properties of graphs we refer the reader to [4].

An additive induced-hereditary property $\mathscr{P}$ is called *reducible* in $\mathbb{M}^a$ if there are non-trivial additive induced-hereditary properties $\mathscr{P}_1, \mathscr{P}_2$ such that $\mathscr{P} = \mathscr{P}_1 \circ \mathscr{P}_2$. An additive hereditary property $\mathscr{P}^*$ is called *reducible* in $\mathbb{L}^a$ if there are non-trivial additive hereditary properties $\mathscr{P}_1^*, \mathscr{P}_2^*$ such that $\mathscr{P}^* = \mathscr{P}_1^* \circ \mathscr{P}_2^*$. If such properties do not exist, the property $\mathscr{P}$ is called *irreducible* in $\mathbb{M}^a$ or *irreducible* in $\mathbb{L}^a$, respectively. Farrugia and Richter [10, Theorem 3.5] proved that each factorization of a reducible additive hereditary property $\mathscr{P}$ into irreducible factors in the lattice $\mathbb{M}^a$ contains only hereditary properties:

Proposition 1. *Let k be a positive integer and $\mathscr{P}$ be an additive hereditary property. Let $\mathscr{Q}_1, \mathscr{Q}_2, \ldots, \mathscr{Q}_l \in \mathbb{M}^a$ be properties of graphs satisfying $\mathscr{P} = \mathscr{Q}_1 \circ \mathscr{Q}_2 \circ \cdots \circ \mathscr{Q}_k$. Then for each $i = 1, 2, \ldots, k$ the property $\mathscr{Q}_i$ belongs to $\mathbb{L}^a$.*

Moreover, Unique Factorization Theorems [10,16,18] yield that the factorization of an additive hereditary (induced-hereditary) property into irreducible factors in $\mathbb{L}^a$ ($\mathbb{M}^a$) is unique (up to the order of factors). Therefore, we need not distinguish between reducibility in $\mathbb{L}^a$ and $\mathbb{M}^a$ and we shall briefly say that $\mathscr{P}$ is reducible or $\mathscr{P}$ is irreducible.

A reducible property $\mathscr{R} \in \mathbb{M}^a$ ($\in \mathbb{L}^a$) is called a *minimal reducible bound* for a property $\mathscr{P} \in \mathbb{M}^a$ ($\in \mathbb{L}^a$) if $\mathscr{P} \subseteq \mathscr{R}$ and there is no reducible property $\mathscr{R}^* \in \mathbb{M}^a$ ($\in \mathbb{L}^a$) satisfying $\mathscr{P} \subseteq \mathscr{R}^* \subset \mathscr{R}$. This means that the obtained colouring result is sharp and in some sense cannot be improved. Finding the set of minimal reducible bounds for given irreducible property is generally a difficult problem, while the only minimal reducible bound that a reducible property $\mathscr{R}$ has is the property itself. The problem of finding all minimal reducible bounds for the class of planar graphs was formulated by Mihók and Toft in 1993 (see Problem 17.9 in [13]). The set of all minimal reducible bounds of a property $\mathscr{P}$ in $\mathbb{M}^a$ and $\mathbb{L}^a$ we denote by $\boldsymbol{B}_M(\mathscr{P})$ and $\boldsymbol{B}_L(\mathscr{P})$, respectively. Some results on minimal reducible bounds can be found in [2,4,5,15,17].

The question on the existence of the set of minimal reducible bounds of a property $\mathcal{P}$ is answered in [1]. There is proved there that

- every additive hereditary property has at least one minimal reducible bound;
- all the reducible bounds of a property $\mathcal{P}$ contain a minimal reducible bound for $\mathcal{P}$;
- every reducible additive hereditary property is a minimal reducible bound for some irreducible additive hereditary property.

The analogous results for the lattice $\mathbb{M}^{\mathrm{a}}$ can be proved by a slight modification of the arguments used in [1].

In Section 2 we prove two important results related to the number and structure of factors of minimal reducible bounds. The main results are proved in Section 3. They provide a method to determine minimal reducible bounds for degenerate properties of graphs. Some applications of the method for new and already known sets of minimal reducible bounds are presented as well. In Section 4 we prove that the set of minimal reducible bounds for k-degenerate graphs in the lattice of induced-hereditary properties is the same as in the lattice of hereditary properties.

2. The structure of sets of minimal reducible bounds

The following straightforward observation will help us to establish the set of minimal reducible bounds for hereditary properties in $\mathbb{L}^{\mathrm{a}}$.

Lemma 2. *Let $\mathcal{P}$ be an additive hereditary property of graphs. If the set of minimal reducible bounds for $\mathcal{P}$ in $\mathbb{M}^{\mathrm{a}}$ consists only of additive hereditary properties then $\mathcal{P}$ has the same set of minimal reducible bounds in $\mathbb{L}^{\mathrm{a}}$ as well.*

The next theorem provides another information on the structure of minimal reducible bounds.

Theorem 3. *Let $\mathcal{P}$ be an additive induced-hereditary (hereditary) property of graphs with $\chi(\mathcal{P})=k$. Then all the minimal reducible bounds for $\mathcal{P}$ are properties consisting of at most k irreducible factors.*

Proof. Since $\chi(\mathcal{P})=k$ there exists a graph $F \notin \mathcal{P}$ such that $\chi(F)=\chi(\mathcal{P})=k$. Let us denote by n the order of the graph F and suppose that there exists a minimal reducible bound $\mathcal{R}$ of $\mathcal{P}$ with the factorization $\mathcal{R}=\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_l$, $l \geqslant k+1$. We shall prove that there exists a reducible additive induced-hereditary (hereditary) property $\mathcal{R}^*$ such that $\mathcal{P}\subseteq\mathcal{R}^*\subset\mathcal{R}$ which contradicts the assumption $\mathcal{R}\in \boldsymbol{B}_M(\mathcal{P})$ ($\mathcal{R}\in\boldsymbol{B}_L(\mathcal{P})$).

Consider the property $\mathcal{R}^*=[(\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k)\cap\mathcal{P}]\circ[\mathcal{P}_{k+1}\circ\mathcal{P}_{k+2}\circ\cdots\circ\mathcal{P}_l]$. Evidently $\mathcal{P}\subseteq\mathcal{R}^*$. On the other hand, since $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_k$ are non-trivial additive hereditary properties of graphs, $\mathcal{O}^k\subseteq\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k$ and it implies that all the k-partite graphs belong to $\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k$.

It follows that $F\in\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k$. Therefore $\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k\backslash\mathcal{P}$ is a non-empty property, $(\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k)\cap\mathcal{P}\subset\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_k$ and, according to the Unique Factorization Theorem (see [18]), the reducible property $(\mathcal{P}_1\circ\cdots\circ\mathcal{P}_k\cap\mathcal{P})\circ\mathcal{P}_{k+1}\circ\cdots\circ\mathcal{P}_l$ is a proper subset of the property $\mathcal{P}_1\circ\mathcal{P}_2\circ\cdots\circ\mathcal{P}_l$. From these facts we obtain that $\mathcal{R}^*\subset\mathcal{R}$ and $\mathcal{R}\notin\boldsymbol{B}_M(\mathcal{P})$ ($\mathcal{R}\notin\boldsymbol{B}_L(\mathcal{P})$), a contradiction. □

Corollary 4. *Any degenerate induced-hereditary (hereditary) property $\mathcal{P}$ has all the minimal reducible bounds of the form $\mathcal{P}_1\circ\mathcal{P}_2$, where $\mathcal{P}_1,\mathcal{P}_2$ are some irreducible degenerate additive induced-hereditary (hereditary) properties.*

Proof. The number of the factors follows from the previous theorem. The degeneracy of the factors follows from the inclusion $\mathcal{P}\subseteq(\mathcal{P}_1\cap\mathcal{P})\circ(\mathcal{P}_2\cap\mathcal{P})$ and from the structure of the set of minimal forbidden graphs of intersection of two induced-hereditary (hereditary) properties (see e.g. [4]). □

3. How to compare different colouring results

The following two results allow us to compare different partitioning or colouring results on an additive induced-hereditary property. Analogous results in the smaller lattice $\mathbb{L}^{\mathrm{a}}$ of additive hereditary properties were proved in [17] and [5], respectively.

Lemma 5. *Let $\mathcal{P},\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3$ and $\mathcal{P}_4$ be additive induced-hereditary properties of graphs satisfying $\mathcal{P}\subseteq\mathcal{P}_1\circ\mathcal{P}_2$ and $\mathcal{P}\xrightarrow{i}(\mathcal{P}_3,\mathcal{P}_4)$. If $\mathcal{P}_3\nsubseteq\mathcal{P}_1$ then $\mathcal{P}_4\subseteq\mathcal{P}_2$.*

Proof. Since $\mathcal{P}_3 \nsubseteq \mathcal{P}_1$ there exists at least one graph $F \in \mathcal{P}_3 \backslash \mathcal{P}_1$. Let H be an arbitrary graph from $\mathcal{P}_4$. Then, according to our assumption $\mathcal{P} \xrightarrow{i} (\mathcal{P}_3, \mathcal{P}_4)$, there is a graph $G \in \mathcal{P}$ such that $G \xrightarrow{i} (F, H)$. But we suppose that $\mathcal{P} \subseteq \mathcal{P}_1 \circ \mathcal{P}_2$ and it implies that there exists a $(\mathcal{P}_1, \mathcal{P}_2)$-partition (V_1, V_2) of the vertex set of G satisfying $G[V_1] \in \mathcal{P}_1$ and $G[V_2] \in \mathcal{P}_2$. As $F \in \mathcal{P}_3 \backslash \mathcal{P}_1$ it cannot appear as an induced subgraph of $G[V_1]$. Therefore we have $H \leqslant G[V_2]$, which implies that H belongs to $\mathcal{P}_2$. Since H was chosen arbitrarily, we immediately have $\mathcal{P}_4 \subseteq \mathcal{P}_2$. □

Theorem 6. *Let $\mathcal{R}_1 \subseteq \mathcal{R}_2$ be additive induced-hereditary degenerate properties and suppose that $\mathcal{P}_1 \circ \mathcal{P}_2 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2$ for some $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{M}^{\mathrm{a}}$. Then either $\mathcal{P}_1 \subseteq \mathcal{R}_1$ and $\mathcal{P}_2 \subseteq \mathcal{R}_2$ or $\mathcal{P}_2 \subseteq \mathcal{R}_1$ and $\mathcal{P}_1 \subseteq \mathcal{R}_2$.*

Proof. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are degenerate properties, there are bipartite graphs F_1, F_2 such that $F_1 \notin \mathcal{R}_1$ and $F_2 \notin \mathcal{R}_2$. By our assumption $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and therefore F_1 can be equal to F_2.

Let us denote by U_1, U_2 the colour classes of a proper 2-colouring of the vertices of $V(F_2)$ (i.e. $V(F_2) = U_1 \cup U_2, U_1 \cap U_2 = \emptyset$ and U_1, U_2 are independent sets of vertices). Let us label the vertices of U_1 by $x_1, x_2, \ldots, x_r$, $r = |U_1|$ and the vertices of U_2 by $y_1, y_2, \ldots, y_s$, $s = |U_2|$.

Suppose now that $\mathcal{P}_1 \nsubseteq \mathcal{R}_1$ and $\mathcal{P}_2 \nsubseteq \mathcal{R}_1$. Then there are graphs $H_1 \in \mathcal{P}_1 \backslash \mathcal{R}_1$ and $H_2 \in \mathcal{P}_2 \backslash \mathcal{R}_1$. Let us construct a new graph G in the following way:

(1) Take r disjoint copies of H_1 and label them by $H_1^1, H_1^2, \ldots, H_1^r$.
(2) Take s disjoint copies of H_2 and label them by $H_2^1, H_2^2, \ldots, H_2^s$.
(3) For each $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, s\}$ and for each $v \in V(H_1^i)$, $u \in V(H_2^j)$ add the edge $\{u, v\}$ whenever $\{x_i, y_j\} \in E(F_2)$.

As $\mathcal{P}_1$ and $\mathcal{P}_2$ are additive properties, it is obvious that $G \in \mathcal{P}_1 \circ \mathcal{P}_2$. Then also $G \in \mathcal{R}_1 \circ \mathcal{R}_2$. Then there is a $(\mathcal{R}_1, \mathcal{R}_2)$-partition (W_1, W_2) of the vertex set $V(G)$ such that $G[W_1] \in \mathcal{R}_1$ and $G[W_2] \in \mathcal{R}_2$. Since $H_1 \notin \mathcal{R}_1, H_2 \notin \mathcal{R}_1$ it is easy to see that at least one vertex of each copy of H_1 and at least one vertex of each copy of H_2 belongs to W_2 (otherwise $H_1 \leqslant G[W_1] \in \mathcal{R}_1$ or $H_2 \leqslant G[W_1] \in \mathcal{R}_1$). But then the set of vertices containing exactly one vertex of each copy of H_1 and H_2 induces a subgraph of $G[W_2]$ isomorphic to $F_2 \notin \mathcal{R}_2$. But this provides a contradiction.

Therefore, $\mathcal{P}_1 \subseteq \mathcal{R}_1$ or $\mathcal{P}_2 \subseteq \mathcal{R}_1$. Without loss of generality we can assume that $\mathcal{P}_1 \subseteq \mathcal{R}_1$. If $\mathcal{P}_2 \nsubseteq \mathcal{R}_2$ then there is a graph $F^* \in \mathcal{P}_2 \backslash \mathcal{R}_2$. Let us put $q = \max\{r, s, |V(F_1)|\}$ and construct a graph G in the following way:

(1) Take $q.2^{q+1}$ disjoint copies of the graph F^* and label them by symbols $F^*_{1,1}, \ldots, F^*_{1,2q}, \ldots, F^*_{2^q,1}, \ldots, F^*_{2^q,2q}$.
(2) Take q independent sets $W_i = \{w_1^i, \ldots, w_q^i\}$, $i = 1, 2, \ldots, q$.
(3) Let $A_1, A_2, \ldots, A_{2^q}$ be the members of the power set of the set $\{1, \ldots, q\}$. For each $i, j, k \in \{1, 2, \ldots, q\}$ and for each $l \in \{1, 2, \ldots, 2^q\}$ join $x \in V(F^*_{l,k})$ to $w_j^i \in W_i$ whenever $j \in A_l$ and $x \in V(F^*_{l,q+k})$ to $w_j^i \in W_i$ whenever $i \in A_l$.

Since $\mathcal{P}_1, \mathcal{P}_2$ are additive properties, we immediately have $G[W_1 \cup \cdots \cup W_q] \in \mathcal{O} \subseteq \mathcal{P}_1$, $G[V(F^*_{1,1}) \cup \cdots \cup V(F^*_{2^q,2q})] \in \mathcal{P}_2$. It follows that $G \in \mathcal{P}_1 \circ \mathcal{P}_2$.

Then also $G \in \mathcal{R}_1 \circ \mathcal{R}_2$. Then there exists a partition (T_1, T_2) of $V(G)$ such that $G[T_1] \in \mathcal{R}_1$ and $G[T_2] \in \mathcal{R}_2$. It is obvious that at least one vertex of each copy of F^* does not belong to T_2 (otherwise $F^* \leqslant G[T_2] \in \mathcal{R}_2$). Moreover, at least one vertex of each copy of F^* belongs to T_2 (otherwise F^* would be an induced subgraph of $G[T_1] \in \mathcal{R}_1 \subseteq \mathcal{R}_2$).

If at least one whole independent set W_i belongs to T_2 then, according to our construction, appropriate number of vertices of W_i together with appropriate vertices from

$$\left(V(F^*_{1,1}) \cup \cdots \cup V(F^*_{1,q}) \cup \cdots \cup V(F^*_{2^q,1}) \cup \cdots \cup V(F^*_{2^q,q})\right) \cap T_2$$

form an induced subgraph of $G[V_2] \in \mathcal{R}_2$ isomorphic to F_2, a contradiction. Therefore at least one vertex of each independent set $W_i, i = 1, 2, \ldots, q$ belongs to T_1. But then appropriate vertices from $\bigcup_{i=1}^q W_i \cap T_1$ and appropriate vertices from

$$\left(V(F^*_{1,q+1}) \cup \cdots \cup V(F^*_{1,2q}) \cup \cdots \cup V(F^*_{2^q,q+1}) \cup \cdots \cup V(F^*_{2^q,2q})\right) \cap T_1$$

evidently induce a subgraph of $G[T_1]$ isomorphic to F_1, which again provides contradiction.

Therefore, $\mathcal{P}_2 \subseteq \mathcal{R}_2$. □

The next theorem provides a method to determine minimal reducible bounds for degenerate induced-hereditary properties of graphs. It is based on the combination of some partitioning and some Ramsey-type results for induced-hereditary properties.

Theorem 7. *Let $\mathcal{O} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$, $k \geqslant 1$ be a chain of additive induced-hereditary degenerate properties of graphs. If for arbitrary non-negative integers r,s,t,u, $r+s+1=k$, $t+u=k$ the properties $\mathcal{P}_r, \mathcal{P}_s, \mathcal{P}_t, \mathcal{P}_u$ satisfy the following two conditions*:

(i) $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s$;
(ii) $\mathcal{P}_k \xrightarrow{i} (\mathcal{P}_t, \mathcal{P}_u)$,

then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $\mathbb{M}^{\mathrm{a}}$ is of the form $\boldsymbol{B}_M(\mathcal{P}_k) = \{\mathcal{P}_p \circ \mathcal{P}_q : p+q+1=k\}$.

Proof. (1) Condition (i) implies that each reducible property $\mathcal{P}_r \circ \mathcal{P}_s$ is a reducible bound for the property $\mathcal{P}_k = \mathcal{P}_{r+s+1}$.

(2) Now we shall show that every reducible property $\mathcal{P}_r \circ \mathcal{P}_s$ is a minimal reducible bound for $\mathcal{P}_k = \mathcal{P}_{r+s+1}$, i.e. there is no reducible property in the interval $(\mathcal{P}_{r+s+1}, \mathcal{P}_r \circ \mathcal{P}_s)$ of the lattice $\mathbb{M}^{\mathrm{a}}$. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be additive induced-hereditary properties such that $\mathcal{P}_{r+s+1} \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2 \subseteq \mathcal{P}_r \circ \mathcal{P}_s$. From Theorem 6 it follows that either $\mathcal{Q}_1 \subseteq \mathcal{P}_r$ and $\mathcal{Q}_2 \subseteq \mathcal{P}_s$ or $\mathcal{Q}_2 \subseteq \mathcal{P}_r$ and $\mathcal{Q}_1 \subseteq \mathcal{P}_s$. Without loss of generality we can assume that $\mathcal{Q}_1 \subseteq \mathcal{P}_r$ and $\mathcal{Q}_2 \subseteq \mathcal{P}_s$.

Suppose now that $\mathcal{Q}_1 \subset \mathcal{P}_r$ (i.e. $\mathcal{P}_r \nsubseteq \mathcal{Q}_1$). Since by our assumption $\mathcal{P}_{r+s+1} \xrightarrow{i} (\mathcal{P}_r, \mathcal{P}_{s+1})$, the application of Lemma 5 yields that $\mathcal{P}_{s+1} \subseteq \mathcal{Q}_2$. But it contradicts the fact $\mathcal{Q}_2 \subseteq \mathcal{P}_s$. Therefore $\mathcal{Q}_1 = \mathcal{P}_r$ and in analogous way we can obtain that $\mathcal{Q}_2 = \mathcal{P}_s$.

(3) It remains to show that for an arbitrary non-negative integer k and any reducible property $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2$, $\mathcal{P}_k \subseteq \mathcal{R}$ there exist $r,s \in \{0,1,2,\ldots\}$ such that $r+s+1=k$ and $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2$. Let j be an integer such that $0 \leqslant j \leqslant k$. By assumption (ii) we have: $\mathcal{P}_k \xrightarrow{i} (\mathcal{P}_j, \mathcal{P}_{k-j})$. From Lemma 5 it follows that $\mathcal{P}_j \subseteq \mathcal{Q}_1$ or $\mathcal{P}_{k-j} \subseteq \mathcal{Q}_2$. If $\mathcal{P}_{k-1} \subseteq \mathcal{Q}_1$ then evidently $\mathcal{P}_{k-1} \circ \mathcal{P}_0 \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2$. Otherwise let $t = \min\{i : \mathcal{P}_i \nsubseteq \mathcal{Q}_1\}$. Note that $t > 0$ because of $\mathcal{O} \subseteq \mathcal{P}$ for every $\mathcal{P} \in \mathbb{M}^{\mathrm{a}}$. Since $t \leqslant k-1$ we have $\mathcal{P}_{t-1} \subseteq \mathcal{Q}_1$ and $\mathcal{P}_{k-t} \subseteq \mathcal{Q}_2$. Hence in both cases the inclusions $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s \subseteq \mathcal{Q}_1 \circ \mathcal{Q}_2$ holds for some r,s satisfying $r+s+1=k$. $\square$

Remark 8. Chains which are mentioned in the previous theorem are very often associated with some monotone additive graph theoretical invariant $\rho(G)$ in the following way:

$$\mathcal{P}_k = \{G \in \mathcal{I} : \rho(G) \leqslant k\}, \quad k = 0,1,2,\ldots.$$

More details can be found in [4,6].

One can immediately see that the following theorem related to the lattice $\mathbb{L}^{\mathrm{a}}$ can be proved in the analogous way as the previous theorem. The only difference is that we have to apply Theorem 3 of [5] and Lemma 2 of [17] instead of Theorem 6 and Lemma 5.

Theorem 9. *Let $\mathcal{O} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$, $k \geqslant 1$ be a chain of additive hereditary degenerate properties of graphs. If for arbitrary non-negative integers r,s,t,u, $r+s+1=k$, $t+u=k$ the properties $\mathcal{P}_r, \mathcal{P}_s, \mathcal{P}_t, \mathcal{P}_u$ satisfy the following two conditions*:

(i) $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s$,
(ii) $\mathcal{P}_k \rightarrow (\mathcal{P}_t, \mathcal{P}_u)$,

then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $\mathbb{L}^{\mathrm{a}}$ is of the form $\boldsymbol{B}_L(\mathcal{P}_k) = \{\mathcal{P}_p \circ \mathcal{P}_q : p+q+1=k\}$.

Here we present some examples of an utilization of the previous theorem. In terms of the described method we determine two new sets of minimal reducible bounds and show that also two already known sets can be obtained in the same manner.

Theorem 10. *For an arbitrary positive integer k the set of minimal reducible bounds of the property $\mathcal{O}_k$ has the following form $\boldsymbol{B}_L(\mathcal{O}_k) = \{\mathcal{O}_p \circ \mathcal{O}_q : p+q+1=k\}$.*

Proof. The properties $\mathcal{O}_k$, $k \geqslant 0$ are uniquely determined by the monotone graph theoretical invariant—number of vertices in a component of a graph. The star $K_{1,k+1}$ does not belong to $\mathcal{O}_k$ and therefore $\mathcal{O}_k$ is degenerate. One can easily observe (see also [4]) that $\mathcal{O}_{p+q+1} \subseteq \mathcal{O}_p \circ \mathcal{O}_q$. Moreover, for an arbitrary choice of $G \in \mathcal{O}_p, H \in \mathcal{O}_q$ (p,q are non-negative integers) there holds $K_{p+q+1} \rightarrow (G,H)$. Therefore $\mathcal{O}_{p+q} \rightarrow (\mathcal{O}_p \circ \mathcal{O}_q)$. Hence using Theorem 9 we obtain $\boldsymbol{B}_L(\mathcal{O}_k) = \{\mathcal{O}_p \circ \mathcal{O}_q : p+q+1=k\}$. $\square$

Theorem 11 (Mihók [17]). *For an arbitrary positive integer k the set of minimal reducible bounds of the property $\mathcal{D}_k$ is of the following form $\boldsymbol{B}_L(\mathcal{D}_k) = \{\mathcal{D}_p \circ \mathcal{D}_q : p+q+1=k\}$.*

Proof. The chain $\mathcal{O} = \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_3 \subseteq \cdots$ is associated with monotone graph theoretical invariant $\rho(G) = \max\{\delta(H) : H \subseteq G\}$. One can easily see that for any positive integer k, the complete bipartite graph $K_{k+1,k+1}$ does not belong to $\mathcal{D}_k$. Therefore $\mathcal{D}_k$ is a degenerate property. Example 1 of [17] shows that $\mathcal{D}_{p+q+1} \subseteq \mathcal{D}_p \circ \mathcal{D}_q$. Theorem 3 of [17] states that $\mathcal{D}_{r+s} \to (\mathcal{D}_r, \mathcal{D}_s)$. Hence by an application of Theorem 9 we obtain $\boldsymbol{B}_L(\mathcal{D}_k) = \{\mathcal{D}_p \circ \mathcal{D}_q : p+q+1 = k\}$. □

Theorem 12. *For an arbitrary positive integer k the set of minimal reducible bounds of the property $\mathcal{P}T_k$ has the following form* $\boldsymbol{B}_L(\mathcal{P}T_k) = \{\mathcal{P}T_p \circ \mathcal{P}T_q : p+q+1 = k\}$.

Proof. The chain $\mathcal{O} = \mathcal{P}T_0 \subseteq \mathcal{P}T_1 \subseteq \mathcal{P}T_2 \subseteq \mathcal{P}T_3 \subseteq \cdots$ is associated with the well-known graph theoretical parameter called tree-width. It is easy to see that for any positive integer k, the complete bipartite graph $K_{k+1,k+1}$ does not belong to $\mathcal{P}T_k$. Therefore $\mathcal{P}T_k$ is degenerate property. Furthermore, Theorem 2.1 of [9] in fact means that $\mathcal{P}T_{p+q+1} \subseteq \mathcal{P}T_p \circ \mathcal{P}T_q$. Using our notation Theorem 3.2 of [9] can be restated in the following way: $\mathcal{P}T_{r+s} \to (\mathcal{P}T_r, \mathcal{P}T_s)$. Hence Theorem 9 yields that $\boldsymbol{B}_L(\mathcal{P}T_k) = \{\mathcal{P}T_p \circ \mathcal{P}T_q : p+q+1 = k\}$. □

The previous three theorems illustrate an application of Theorem 9 for infinite chains of additive hereditary properties defined by a standard monotone graph theoretical invariant. The following result provides an application of Theorem 9 to a different type of chain.

Theorem 13 (Mihók [15]). $\boldsymbol{B}_L(\mathcal{T}_2) = \{\mathcal{O} \circ \mathcal{D}_1, (\mathcal{D}_1 \cap \mathcal{S}_2)^2\}$.

Proof. Let us consider the chain $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3$ where $\mathcal{P}_0 = \mathcal{O}$, $\mathcal{P}_1 = \mathcal{D}_1 \cap \mathcal{S}_2$, $\mathcal{P}_2 = \mathcal{D}_1$, $\mathcal{P}_3 = \mathcal{T}_2$. It is known (for details see [15]) that $\mathcal{P}_3 = \mathcal{T}_2 \subseteq \mathcal{O} \circ \mathcal{D}_1 = \mathcal{P}_0 \circ \mathcal{P}_2$ and $\mathcal{P}_3 = \mathcal{T}_2 \subseteq (\mathcal{D}_1 \circ \mathcal{S}_2)^2 = \mathcal{P}_1 \circ \mathcal{P}_1$.

Moreover, it is trivial that $\mathcal{P}_3 \to (\mathcal{P}_0, \mathcal{P}_3)$, i.e. $\mathcal{T}_2 \to (\mathcal{O}, \mathcal{T}_2)$. For an application of Theorem 9 we need also the following arrow relation $\mathcal{P}_3 \to (\mathcal{P}_1, \mathcal{P}_2)$, which in fact means $\mathcal{T}_2 \to (\mathcal{D}_1 \cap \mathcal{S}_2, \mathcal{D}_1)$. The relation $\mathcal{P}_3 \to (\mathcal{P}_2, \mathcal{P}_1)$ is symmetrical. One can easily see that, in order to prove the relation $\mathcal{T}_2 \to (\mathcal{D}_1 \cap \mathcal{S}_2, \mathcal{D}_1)$, it is sufficient to consider only the paths (the edge maximal graphs from $\mathcal{D}_1 \cap \mathcal{S}_2$) and the trees (the edge maximal graphs from $\mathcal{D}_1$). Hence, by the construction from the proof of Lemma 2 in [15], we obtain a graph satisfying that for an arbitrary its vertex colouring with red and blue either the given path is subgraph of red colour or the prescribed tree is a subgraph of blue colour. Therefore the relations $\mathcal{D}_2 \to (\mathcal{D}_1 \cap \mathcal{S}_2, \mathcal{D}_1)$, $\mathcal{D}_2 \to (\mathcal{D}_1, \mathcal{D}_1 \cap \mathcal{S}_2)$ are valid too.

Finally, by an application of Theorem 9 we obtain $\boldsymbol{B}_L(\mathcal{T}_2) = \{\mathcal{O} \circ \mathcal{D}_1, (\mathcal{D}_1 \cap \mathcal{S}_2)^2\}$. □

4. Minimal reducible bounds for *k*-degenerate graphs

The sets of minimal reducible bounds for k-degenerate graphs in the lattice $\mathbb{L}^{\mathrm{a}}$ are determined in [17]. Here we shall prove that the set of minimal reducible bounds of k-degenerate graphs in the lattice $\mathbb{M}^{\mathrm{a}}$ is the same. To apply our method developed in Section 3, we need some Ramsey-type results.

Lemma 14. *Let $\mathcal{P}$ be an additive induced-hereditary property of graphs. Then* $\mathcal{P} \xrightarrow{i} (\mathcal{O}, \mathcal{P})$.

Proof. Let G be an arbitrary graph belonging to $\mathcal{P}$ and D_n be the edgeless graph of order n. We claim that the graph $H = nG$ (disjoint union of n copies of G) satisfies $H \xrightarrow{i} (D_n, G)$.

Evidently, for an arbitrary colouring of vertices of H with blue and red either there is a blue copy of G in the graph H (then $G \leqslant H$) or at least one vertex of each copy of G is red. Then we choose one red vertex from each copy and the obtained set of vertices induces subgraph of H isomorphic to D_n. □

Theorem 15. *Let p, q be non-negative integers. Then* $\mathcal{D}_{p+q} \xrightarrow{i} (\mathcal{D}_p, \mathcal{D}_q)$.

Proof. Without loss of generality, we can assume that $0 \leqslant p \leqslant q$. If $p = 0$ then $\mathcal{D}_p = \mathcal{O}$ and according to Lemma 14 we have $\mathcal{D}_q \xrightarrow{i} (\mathcal{O}, \mathcal{D}_q)$.

Now, we want to prove that for an arbitrary graph $H_1 \in \mathcal{D}_p$ and an arbitrary graph $H_2 \in \mathcal{D}_q$ there exists a graph $G \in \mathcal{D}_{p+q}$ such that $G \xrightarrow{i} (H_1, H_2)$. We shall proceed by induction on $|V(H_1)|$.

(1) If $H_1 = K_1$ then $G = H_2$ since $H_2 \xrightarrow{i} (K_1, H_2)$ and evidently $H_2 \in \mathcal{D}_{p+q}$.

(2) Suppose that for a positive integer $s \geqslant 1$ and an arbitrary p-degenerate graph H_1 of order s and an arbitrary q-degenerate graph H_2 there is a graph $G \in \mathcal{D}_{p+q}$ such that $G \xrightarrow{i} (H_1, H_2)$. Now, let $H_1 \in \mathcal{D}_p$ be a graph of order

$s+1$ and $H_2 \in \mathscr{D}_q$. Since $H_1 \in \mathscr{D}_p$, there is a vertex $w \in V(H_1)$ of degree at most p such that $H_1 - w \in \mathscr{D}_p$. Then by the induction hypothesis, there is a graph G^* such that $G^* \xrightarrow{i} (H_1 - w, H_2)$. Let us denote by $\mathscr{N}$ the neighbourhood $N_{H_1}(w)$ of w in H_1.

Let us label vertices of G^* and let $\mathscr{A} = \{S_i \subseteq V(G^*) : 1 \leqslant |S_i| \leqslant p,\ i = 1,2,\ldots,t\}$, $t = \sum_{i=1}^{p} \binom{|V(G^*)|}{i}$, be the set of all non-empty subsets of $V(G^*)$ with at most p elements. Evidently $\mathscr{N} \in \mathscr{A}$. For any graphs G, H and $S \subseteq V(G)$ we denote by $\Omega(G,H,S)$, the graph obtained from G by joining all the vertices of H to each vertex of S. Thus the graph $\Omega(G,H,S)$ has order $|V(G)| + |V(H)|$ and if $G \in \mathscr{D}_{p+q}, H \in \mathscr{D}_q, |S| \leqslant p$ then $\Omega(G,H,S) \in \mathscr{D}_{p+q}$. Moreover, if $F \leqslant G$ then $F \leqslant \Omega(G,H,S)$.

Let $M_0 = G^*$ and $M_i = \Omega(M_{i-1}, H_2^i, S_i)$, where $H_2^i = H_2$, $S_i \in \mathscr{A}$, $i = 1,2,\ldots,t$ and $\bigcup_{i=1}^{t} S_i = \infty$. Evidently M_t is $p+q$-degenerate. Let us colour the graph M_t with red and blue. According to our assumption either there is a blue induced subgraph of G^* (we remind that $G^* \leqslant M_t$) isomorphic to H_2 or a red induced subgraph of G^* isomorphic to $H_1 - w$. In the first case there is nothing more to do. In the second case consider the colouring of the vertices of $H_2^1, H_2^2, \ldots, H_2^t$. Either at least one of these graphs is blue and H_2 is and induced subgraph of M_t and we have done, or at least one vertex of each graph $H_2^1, H_2^2, \ldots, H_2^t$ is red. According to our construction there exist an $1 \leqslant j \leqslant t$ such that $S_j = \mathscr{N}$. Therefore we can choose a red vertex belonging to a $V(H_2^j)$ which together with the induced red copy of $H_1 - w$ in G^* forms an induced copy of H_1 in red colour. Hence $M_t \xrightarrow{i} (H_1, H_2)$ and the proof is complete. □

By an application of Theorem 7 and the previous results we immediately have the following:

Theorem 16. *For an arbitrary positive integer k the following holds*

$$\mathbf{B}_M(\mathscr{D}_k) = \{\mathscr{D}_p \circ \mathscr{D}_q : p + q + 1 = k\}.$$

According to Lemma 2 one can immediately see that our previous theorem imply Theorem 11.

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References

[1] A.J. Berger, Minimal reducible bounds in the lattice of additive hereditary graph properties, Discrete Math. 251 (1–3) (2002) 3–10.
[2] A.J. Berger, I. Broere, Minimal reducible bounds for hom-properties of graphs, Discuss. Math. Graph Theory 19 (1999) 143–158.
[3] B. Bollobás, A.G. Thomason, Hereditary and monotone properties of graphs, in: R.L. Graham, J. Nešetřil (Eds.), The Mathematics of Paul Erdős, II, Algorithms and Combinatorics, Vol. 14, Springer, Berlin, 1997, pp. 70–78.
[4] M. Borowiecki, I. Broere, M. Frick, P. Mihók, G. Semanišin, Survey of hereditary properties of graphs, Discuss. Math. Graph Theory 17 (1997) 5–50.
[5] M. Borowiecki, I. Broere, P. Mihók, Minimal reducible bounds for planar graphs, Discrete Math. 212 (2000) 19–27.
[6] S.A. Burr, M.S. Jacobson, P. Mihók, G. Semanišin, Generalized Ramsey theory and decomposable properties of graphs, Discuss. Math. Graph Theory 19 (1999) 199–217.
[7] G. Chartrand, D. Geller, S. Hedetniemi, Graphs with forbidden subgraphs, J. Combin. Theory B 10 (1971) 12–41.
[8] E.J. Cockayne, Color classes for r-graphs, Canad. Math. Bull. 15 (1972) 349–354.
[9] G. Ding, B. Oporowski, D.P. Sanders, D. Vertigan, Partitioning graphs of bounded tree-width, Combinatorica 18 (1998) 1–12.
[10] A. Farrugia, R.B. Richter, Unique factorisation of additive induced-hereditary properties, Discuss. Math. Graph Theory 24 (2004) 319–343.
[11] M. Frick, A survey of (m,k)-colorings, in: J. Gimbel c.a (Ed.), Quo Vadis, Graph Theory? A source book for challenges and directions, Annals of Discrete Mathematics, Vol. 55, North-Holland, Amsterdam, 1993, pp. 45–57.
[12] S.T. Hedetniemi, On hereditary properties of graphs, J. Combin. Theory B 14 (1973) 94–99.
[13] T.R. Jensen, B. Toft, Graph Colouring Problems, Wiley–Interscience Publications, New York, 1995.
[14] P. Mihók, Additive hereditary properties and uniquely partitionable graphs, in: M. Borowiecki, Z. Skupien (Eds.), Graphs, Hypergraphs and Matroids, Wysza Szkola Inzynierska Zielona Góra, 1985, pp. 49–58.
[15] P. Mihók, On the minimal reducible bound for outerplanar and planar graphs, Discrete Math. 150 (1996) 431–435.
[16] P. Mihók, Unique factorization theorem, Discuss. Math. Graph Theory 20 (2000) 143–154.

[17] P. Mihók, Minimal reducible bounds for the class of k-degenerate graphs, Graph Theory, Kazimierz Dolny, 1997, Discrete Math. 236 (1–3) (2001) 273–279.
[18] P. Mihók, G. Semanišin, R. Vasky, Additive and hereditary properties of graphs are uniquely factorizable into irreducible factors, J. Graph Theory 33 (2000) 44–53.
[19] J. Mitchem, Maximal k-degenerate graphs, Utilitas Math. 11 (1977) 101–106.